Introduction to Reliability and Sensitivity Analysis

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Outline

- Formulation of structural reliability problem
- Solution methods
- Uncertainty propagation
- Response sensitivity analysis
- Sensitivity/importance measures
- Methods implemented in OpenSees
- Example – probabilistic pushover analysis
- Stochastic nonlinear dynamic analysis
- Example – fragility analysis of hysteretic system
- Summary and conclusions
Formulation of structural reliability

\[ \mathbf{X} = \left\{ \begin{array}{c} X_1 \\ \vdots \\ X_n \end{array} \right\} \]  
vector of random variables

\[ f_X(x) \]  
distribution of \( \mathbf{X} \)

\[ \Omega_x \]  
failure domain

\[ p_f = \int_{\Omega_x} f_X(x) \, dx \]  
failure probability
Formulation of structural reliability

Component reliability problem

\[ \Omega_x \equiv \{ g(x) \leq 0 \} \]

System reliability problem

\[ \Omega_x \equiv \left\{ \bigcup_{k \in C_k} \bigcap_{i \in C_k} g_i(x) \leq 0 \right\} \]

\( g(x), \ g_i(x) \) limit-state functions (must be continuously differentiable)

e.g., \( g_i(x) = \delta_{cr} - \delta_i(x) \) failure due to excessive \( i^{th} \) story drift
Solution by First-Order Reliability Method (FORM)

\[ \mathbf{u} = \mathbf{T}(\mathbf{x}), \quad g(\mathbf{x}) \rightarrow G(\mathbf{u}) \]

\[ \mathbf{u}^* = \arg \min \limits_{\mathbf{u}} \left\{ \| \mathbf{u} \| \mid G(\mathbf{u}) \leq 0 \right\} \]

\[ \beta = \alpha \cdot \mathbf{u}^* \quad \text{reliability index} \]

\[ \alpha = -\frac{\nabla G(\mathbf{u})}{\|\nabla G(\mathbf{u})\|}_{\mathbf{u} = \mathbf{u}^*} \]

\[ p_f \approx \Phi(-\beta) \]

Monte Carlo simulation

\[
p_f \approx \frac{1}{N} \sum_{i=1}^{N} I\{G(u_i) \leq 0\}
\]

\[
I\{G(u_i) \leq 0\} = 1 \text{ if } G(u_i) \leq 0
\]

\[
= 0 \text{ otherwise}
\]

\[u_i\] simulated according to \(\varphi(u)\)
Solution by importance sampling

\[ p_f \approx \frac{1}{N} \sum_{i=1}^{N} I\{G(u_i) \leq 0\} \frac{(u_i)}{h(u_i)} \]

\( u_i \) simulated according to \( h(u) \)
Uncertainty propagation

\[ \delta = \delta(X) \text{ response quantity of interest} \]

First-order approximations:

\[ \mu_\delta \approx \delta(M_X) \]

\[ \sigma_\delta^2 \approx \nabla_x \delta \Sigma_{xx} \nabla_x \delta^T \]

\[ M_X \text{ mean vector} \]

\[ \Sigma_{xx} \text{ covariance matrix} \]

\[ \nabla_x \delta = \begin{bmatrix} \frac{\partial \delta}{\partial x_1} & \cdots & \frac{\partial \delta}{\partial x_n} \end{bmatrix} \text{ gradient row-vector} \]
Response sensitivity analysis

For both FORM and FOSM, we need $\frac{\partial \delta(x)}{\partial x_i}$

Available methods in OpenSees:

- **Finite difference**

- **Direct Differentiation Method (DDM)**
  Differentiate equations of motion and solve for response derivative equations as adjoint to the equations of motion. Equations for the response derivative are linear, even for nonlinear response.

  DDM is more stable, accurate and efficient than finite difference.


Brief on DDM

Linear static problems:

\[ K(x)\delta(x) = P(x) \]

\[ \frac{\partial K}{\partial x} \delta + K \frac{\partial \delta}{\partial x} = \frac{\partial P}{\partial x} \]

\[ \frac{\partial \delta}{\partial x} = K^{-1} \left( \frac{\partial P}{\partial x} - \frac{\partial K}{\partial x} \delta \right) \]
Brief on DDM

Nonlinear static problems:

\[ R(\delta(x), x) = P(x) \]

\[ \frac{\partial R(\delta, x)}{\partial \delta} \frac{\partial \delta}{\partial x} + \frac{\partial R(\delta, x)}{\partial x} = \frac{\partial P}{\partial x} \]

\[ \frac{\partial \delta}{\partial x} = K_t^{-1} \left( \frac{\partial P}{\partial x} - \frac{\partial R(\delta, x)}{\partial x} \right) \]

\[ K_t^{-1} = \frac{\partial R(\delta, x)}{\partial \delta} = \text{tangent stiffness matrix} \]

\[ \frac{\partial R(\delta, x)}{\partial x} = \sum_e \int \mathbf{B}^T(v) \frac{\partial \sigma(v, x)}{\partial x} \bigg|_{\text{fixed } \varepsilon} \ d\Omega \]
Sensitivity/Importance measures

FOSM: \( \frac{\partial \delta(x)}{\partial x_i} \sigma_i \)

FORM: 
- \( \alpha = \text{relative importance of } u \text{ variables} \)
- \( \gamma = \text{relative importance of } X \text{ variables} \)
- \( \delta = \left\{ \frac{\partial \beta}{\partial \mu_i} \sigma_i \right\} = \text{reliability importance of mean values} \)
- \( \eta = \left\{ \frac{\partial \beta}{\partial \sigma_i} \sigma_i \right\} = \text{reliability importance of stdev values} \)

Methods implemented in OpenSees

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<td>Estimate probability of events defined in terms of limit-state functions</td>
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<td>Response sensitivity analysis:</td>
<td>The Direct Differentiation Method (DDM)</td>
<td>( \frac{\partial \text{response}}{\partial \text{parameter}} )</td>
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<td>Determine derivative of response with respect to input or structural properties</td>
<td>Finite Difference scheme (FD)</td>
<td>(Used in FOSM and FORM analysis)</td>
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The I-880 Testbed Bridge

320 random variables

\[ g_1 = u_o - u(\lambda_o) \]
\[ g_2 = \lambda_o - \lambda(u_o) \]
\[ g_3 = u(20\% \text{ tangent}) - u_o \]
\[ g_4 = \lambda(20\% \text{ tangent}) - \lambda_o \]
First-Order Second-Moment Analysis

Second-moment response statistics for $u(\lambda_0)$

Second-moment response statistics for $\lambda(u_0)$
FORM Analysis, $g_1$, $\lambda_0 = 0.20$
\[ g_1 = 0.35 - u(\lambda=0.2) \]
FORM Analysis, $g_2, \ u_o=0.30$
Continuity of response derivative
FORM Analysis, $g_3$ and $g_4$

$g_3 = u(20\% \text{ tangent}) - u_o$

$g_4 = \lambda(20\% \text{ tangent}) - \lambda_o$

Stochastic nonlinear dynamic analysis

Representation of stochastic ground motion

\[ A(t, u) = q(t) \sum_{i=1}^{n} s_i(t) u_i = q(t) s(t) u \]

- Stochasticity
- Temporal nonstationarity
- Spectral nonstationarity

\[ q(t_i)u_i \quad s_i(t) \quad \text{normalized response of a time-dependent filter} \]

\[ t_{i-1} \quad t_i \quad t_{i+1} \quad t \]
Stochastic nonlinear dynamic analysis

Representation of stochastic ground motion

\[ A(t, u) = q(t) \sum_{i=1}^{n} s_i(t)u_i = q(t)s(t)u \]

Stochastic nonlinear dynamic analysis

\[ \Pr[x \leq X(t, u)] \]  

tail probability for threshold \( x \) at time \( t \)

Solution of the above problem by FORM leads to identification of a Tail-Equivalent Linear System (TELS).

TELS is solved by linear random vibration methods to obtain response statistics of interest, e.g., distribution of extreme peak response, fragility curve.

Application to MDOF hysteretic system

Node 6
\[ m = 3.0 \times 10^4 \text{kg for all nodes} \]
\[ k_0 = 2.0 \times 10^4 \text{kN/m} \]

Node 5
\[ k_0 = 4.0 \times 10^4 \text{kN/m} \]

Node 4
\[ k_0 = 5.5 \times 10^4 \text{kN/m} \]

Node 3
\[ k_0 = 6.5 \times 10^4 \text{kN/m} \]

Node 2
\[ k_0 = 7.0 \times 10^4 \text{kN/m} \]

Node 1
\[ k_0 = 7.5 \times 10^4 \text{kN/m} \]

Smooth bilinear hysteresis model
Fragility curves for story drifts

Summary and conclusions

- Reliability analysis requires specification of uncertain quantities, their distributions, and definition of performance via limit-state function(s). It provides probability of exceeding specified performance limit(s).

- Uncertainty propagation provides first-order approximation of mean, variance and correlations of response quantities.

- Sensitivity and importance measures provide insight into the relative importance of variables and parameters.

- Stochastic dynamic analysis is performed via tail-equivalent linearization. TELM can be used to generate fragility functions (e.g., in lieu of performing IDAs).