Introduction to Reliability and Sensitivity Analysis

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Outline

- Formulation of structural reliability problem
- Solution methods
- Uncertainty propagation
- Response sensitivity analysis
- Sensitivity/importance measures
- Methods implemented in OpenSees
- Example – probabilistic pushover analysis
- Stochastic nonlinear dynamic analysis
- Example – fragility analysis of hysteretic system
- Summary and conclusions

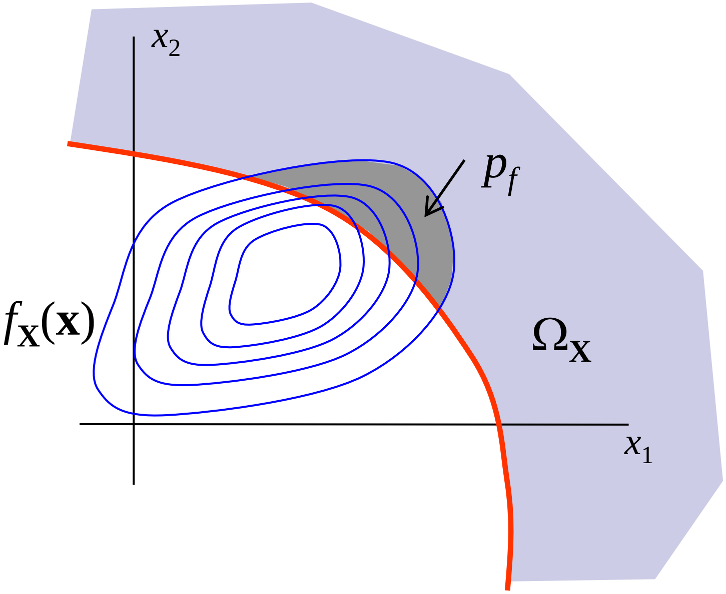
Formulation of structural reliability

$$\mathbf{X} = \begin{Bmatrix} X_1 \\ \vdots \\ X_n \end{Bmatrix}$$ Vector of random variables

$f_{\mathbf{X}}(\mathbf{x})$ Distribution of **X**

$\Omega_{\mathbf{x}}$ Failure domain

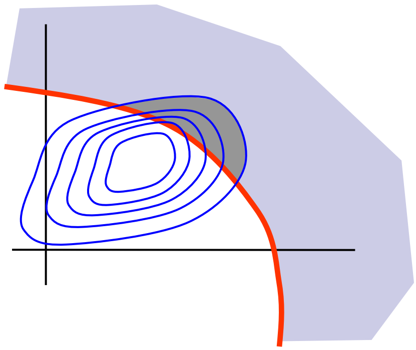
$$p_f = \int_{\Omega_{\mathbf{x}}} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$ Failure probability



Formulation of structural reliability

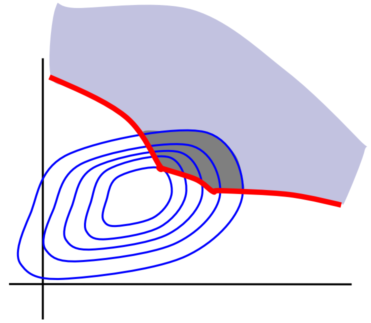
Component reliability problem

$$\Omega_{\mathbf{x}} \equiv \{g(\mathbf{x}) \le 0\}$$



System reliability problem

$$\Omega_{\mathbf{x}} \equiv \left\{ \bigcup_{k} \bigcap_{i \in C_k} g_i(\mathbf{x}) \le 0 \right\}$$



$g(\mathbf{x}),\ g_i(\mathbf{x})$ limit-state functions (must be continuously differentiable)

e.g., $g_i(\mathbf{x}) = \delta_{cr} - \delta_i(\mathbf{x})$ failure due to excessive i^{th} story drift

Solution by First-Order Reliability Method (FORM)

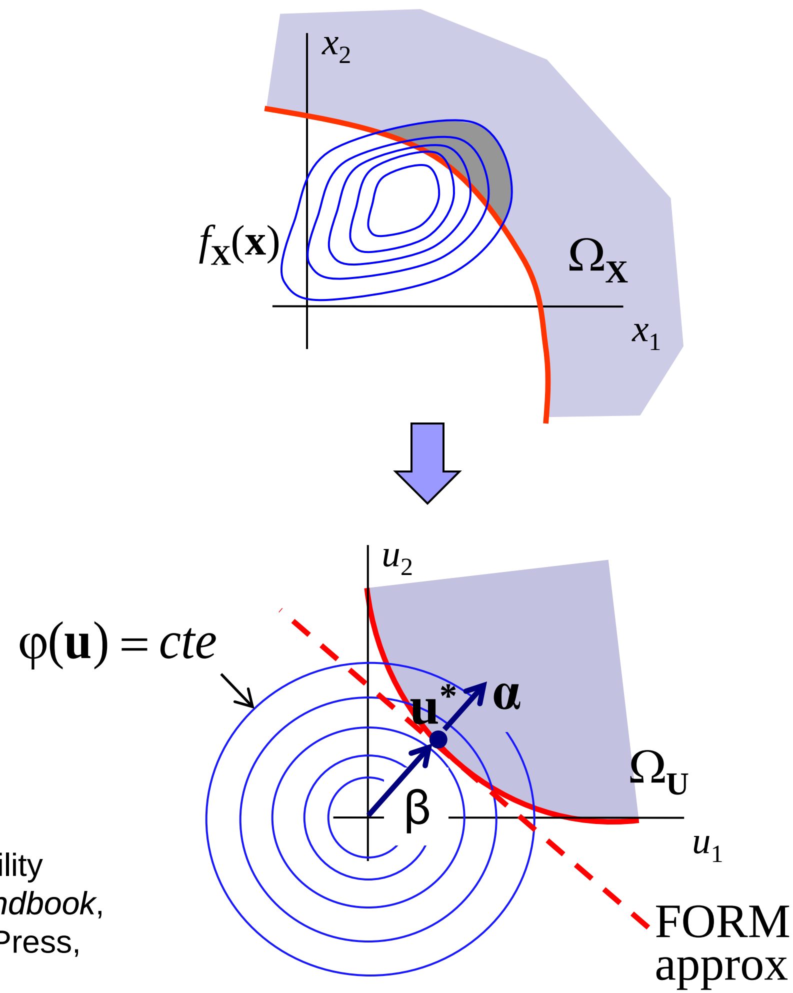
$$\mathbf{u} = \mathbf{T}(\mathbf{x}), \quad g(\mathbf{x}) \to G(\mathbf{u})$$

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \left\{ \|\mathbf{u}\| \;\middle|\; G(\mathbf{u}) \le \mathbf{0} \right\}$$

$$\beta = \boldsymbol{\alpha} \cdot \mathbf{u}^* \quad \text{reliability index}$$

$$\boldsymbol{\alpha} = -\left.\frac{\nabla G(\mathbf{u})}{\|\nabla G(\mathbf{u})\|}\right|_{\mathbf{u}=\mathbf{u}^*}$$

$$p_f \cong \Phi(-\beta)$$



Der Kiureghian, A. (2005). First- and second-order reliability methods. Chapter 14 in *Engineering design reliability handbook*, E. Nikolaidis, D. M. Ghiocel and S. Singhal, Edts., CRC Press, Boca Raton, FL.

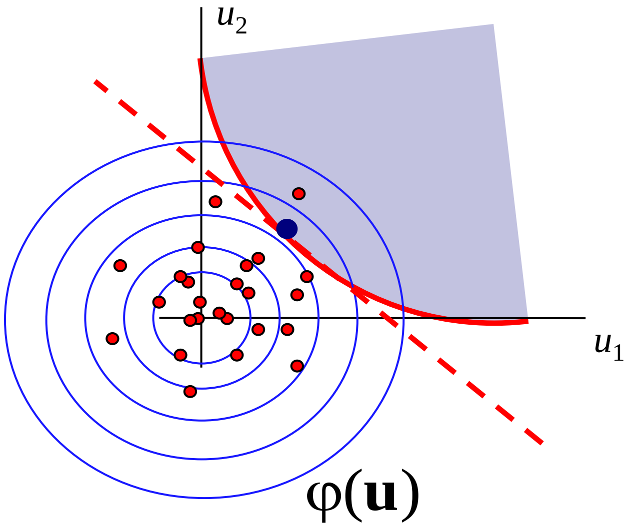
Monte Carlo simulation

$$p_f \cong \frac{1}{N}\sum_{i=1}^{N} I\{G(\mathbf{u}_i) \le 0\}$$

$$I\{G(\mathbf{u}_i) \le 0\} = 1 \text{ if } G(\mathbf{u}_i) \le 0$$

$$= 0 \quad \text{otherwise}$$

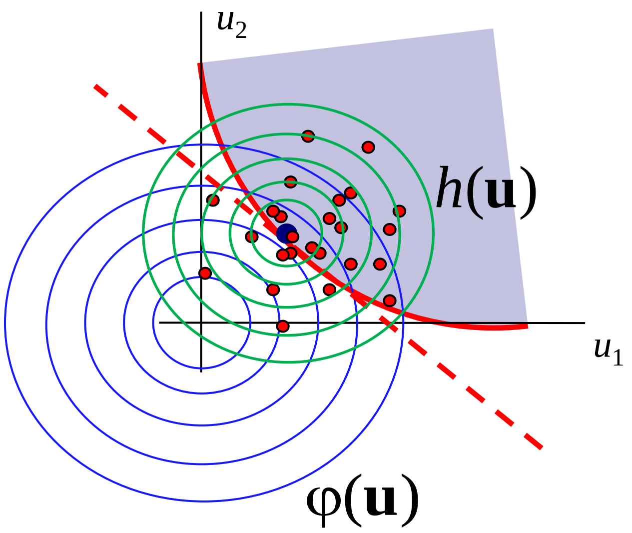
$\mathbf{u}_i$ simulated according to $\varphi(\mathbf{u})$



Solution by importance sampling

$$p_f \cong \frac{1}{N}\sum_{i=1}^{N} I\{G(\mathbf{u}_i) \le 0\}\frac{(\mathbf{u}_i)}{h(\mathbf{u}_i)}$$

$\mathbf{u}_i$ simulated according to $h(\mathbf{u})$



Uncertainty propagation

$\delta = \delta(\mathbf{X})$ response quantity of interest

First-order approximations:

$$\mu_\delta \cong \delta(\mathbf{M}_\mathrm{X})$$

$$\sigma_\delta^2 \cong \nabla_\mathrm{X}\delta\, \boldsymbol{\Sigma}_\mathrm{XX} \nabla_\mathrm{X}\delta^\mathrm{T}$$

$\mathbf{M}_\mathrm{X}$ mean vector

Σ_XX covariance matrix

$\nabla_\mathrm{X}\delta = \begin{bmatrix} \dfrac{\partial\delta}{\partial x_1} & \cdots & \dfrac{\partial\delta}{\partial x_n} \end{bmatrix}$ gradient row-vector

Response sensitivity analysis

For both FORM and FOSM, we need $\dfrac{\partial \delta(\mathbf{x})}{\partial x_i}$

Available methods in OpenSees:

- Finite difference
- Direct Differentiation Method (DDM)

  Differentiate equations of motion and solve for response derivative equations as adjoint to the equations of motion. Equations for the response derivative are linear, even for nonlinear response.

  DDM is more stable, accurate and efficient than finite difference.

- Zhang, Y., and A. Der Kiureghian (1993). Dynamic response sensitivity of inelastic structures. *Comp. Methods Appl. Mech. Engrg*., 108(1), 23-36.
- Haukaas, T., and A. Der Kiureghian (2005). Parameter sensitivity and importance measures in nonlinear finite element reliability analysis. *J. Engineering Mechanics*, ASCE, **131**(10): 1013-1026.

Brief on DDM

Linear static problems:

$$\mathbf{K}(\mathbf{x})\boldsymbol{\delta}(\mathbf{x}) = \mathbf{P}(\mathbf{x})$$

$$\frac{\partial \mathbf{K}}{\partial x}\boldsymbol{\delta} + \mathbf{K}\frac{\partial \boldsymbol{\delta}}{\partial x} = \frac{\partial \mathbf{P}}{\partial x}$$

$$\frac{\partial \boldsymbol{\delta}}{\partial x} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{P}}{\partial x} - \frac{\partial \mathbf{K}}{\partial x}\boldsymbol{\delta}\right)$$

Brief on DDM

Nonlinear static problems:

$$\mathbf{R}(\boldsymbol{\delta}(x), x) = \mathbf{P}(\mathbf{x})$$

$$\frac{\partial \mathbf{R}(\boldsymbol{\delta}, x)}{\partial \boldsymbol{\delta}} \frac{\partial \boldsymbol{\delta}}{\partial x} + \frac{\partial \mathbf{R}(\boldsymbol{\delta}, x)}{\partial x} = \frac{\partial \mathbf{P}}{\partial x}$$

$$\frac{\partial \boldsymbol{\delta}}{\partial x} = \mathbf{K}_t^{-1} \left(\frac{\partial \mathbf{P}}{\partial x} - \frac{\partial \mathbf{R}(\boldsymbol{\delta}, x)}{\partial x} \right)$$

$$\mathbf{K}_t^{-1} = \frac{\partial \mathbf{R}(\boldsymbol{\delta}, x)}{\partial \boldsymbol{\delta}} = \text{ tangent stiffness matrix}$$

$$\frac{\partial \mathbf{R}(\boldsymbol{\delta}, x)}{\partial x} = \sum_e \int \mathbf{B}^T(\mathbf{v}) \left. \frac{\partial \boldsymbol{\sigma}(\mathbf{v}, x)}{\partial x} \right|_{\text{fixed } \boldsymbol{\varepsilon}} d\Omega$$

Sensitivity/Importance measures

FOSM: $$\frac{\partial \delta(\mathbf{x})}{\partial x_i}\sigma_i$$

FORM: $$\boldsymbol{\alpha} = \text{relative importance of } \mathbf{u} \text{ variables}$$

$$\boldsymbol{\gamma} = \text{relative importance of } \mathbf{X} \text{ variables}$$

$$\boldsymbol{\delta} = \left\{ \frac{\partial \beta}{\partial \mu_i}\sigma_i \right\} = \text{reliability importance of mean values}$$

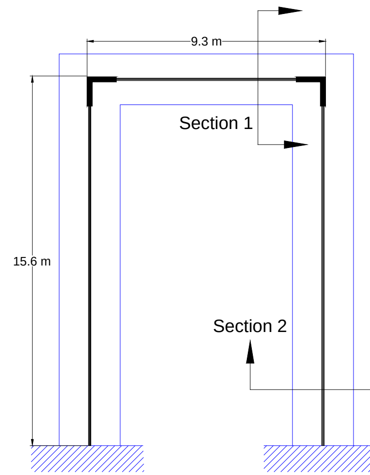
$$\boldsymbol{\eta} = \left\{ \frac{\partial \beta}{\partial \sigma_i}\sigma_i \right\} = \text{reliability importance of stdev values}$$

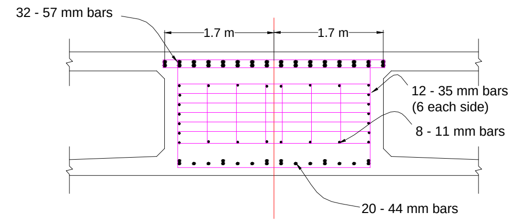
Haukaas, T., and A. Der Kiureghian (2005). Parameter sensitivity and importance measures in nonlinear finite element reliability analysis. *J. Engineering Mechanics*, ASCE, **131**(10): 1013-1026.

Methods implemented in OpenSees

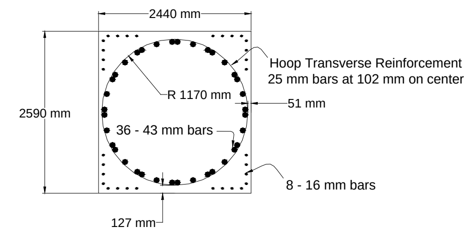
Propagation of uncertainty: *Estimate second moments of response*	First-Order Second-Moment (FOSM) Monte Carlo sampling (MCS)	• Mean • Standard deviation • Correlation • Parameter importance at mean point
Reliability analysis: *Estimate probability of events defined in terms of limit-state functions*	First-Order Reliability Method (FORM) Importance Sampling (IS) Second-Order Reliability Method (SORM)	• "Design point" • Probability of failure • Parameter importance/ sensitivity measures
Response sensitivity analysis: *Determine derivative of response with respect to input or structural properties*	The Direct Differentiation Method (DDM) Finite Difference scheme (FD)	• $\frac{\partial \text{ response}}{\partial \text{ parameter}}$ (Used in FOSM and FORM analysis)

The I-880 Testbed Bridge

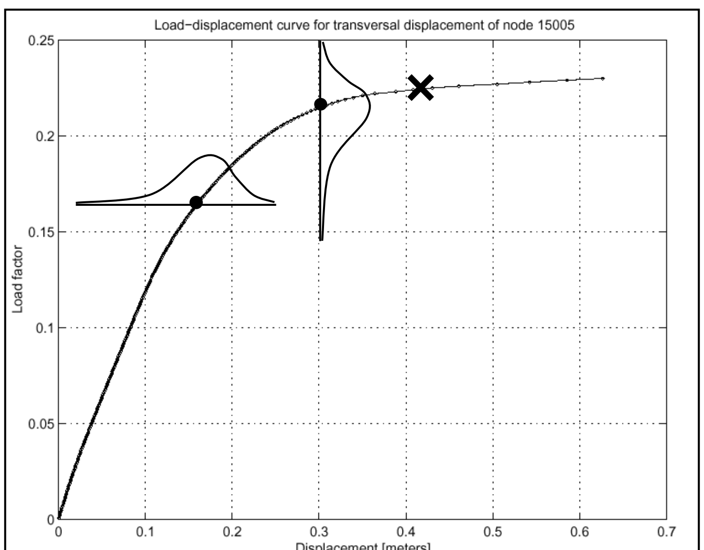




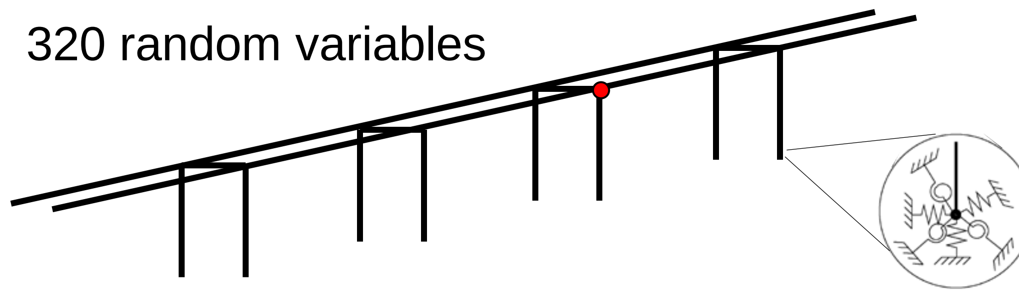
Section 1



Section 2

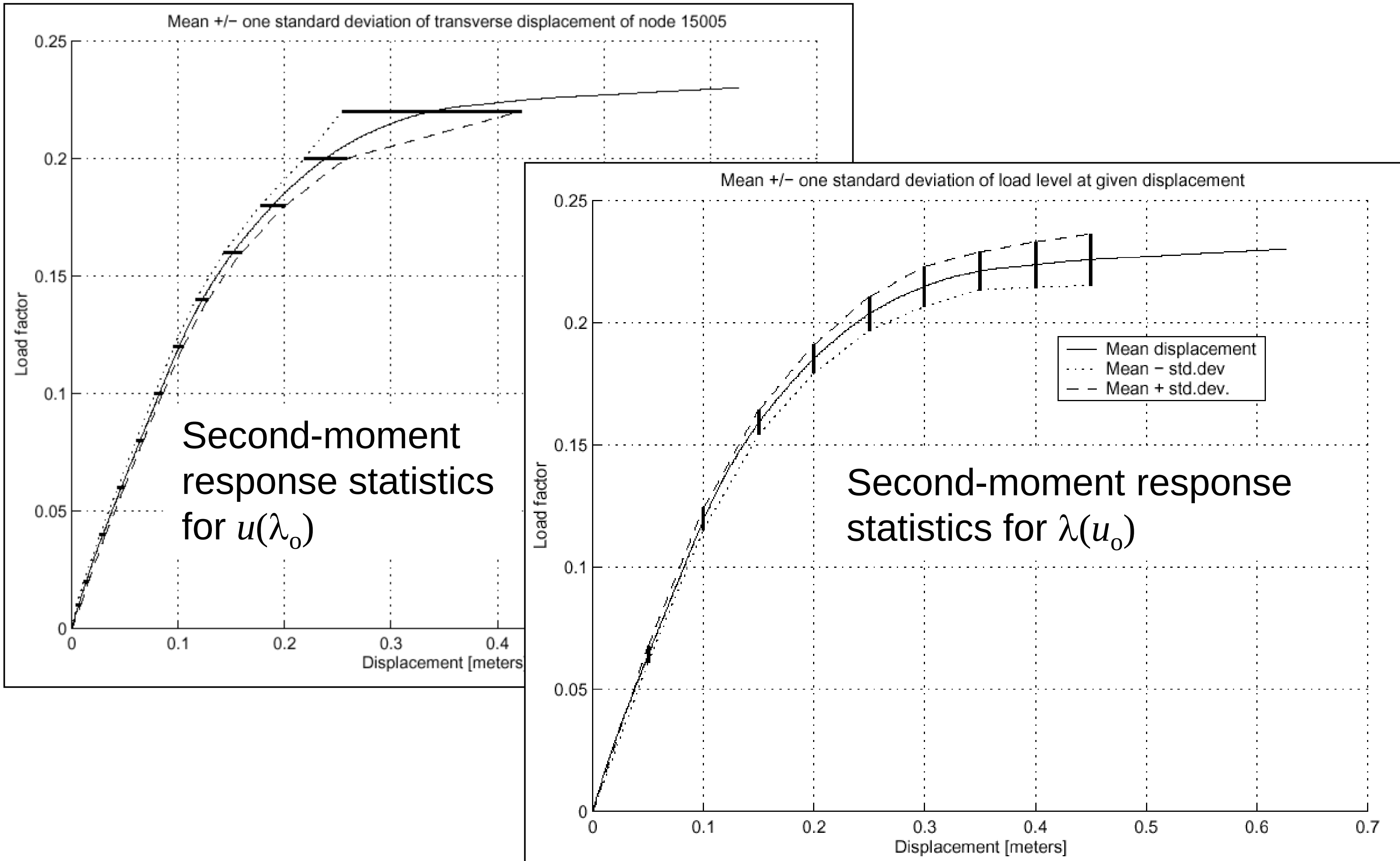


320 random variables

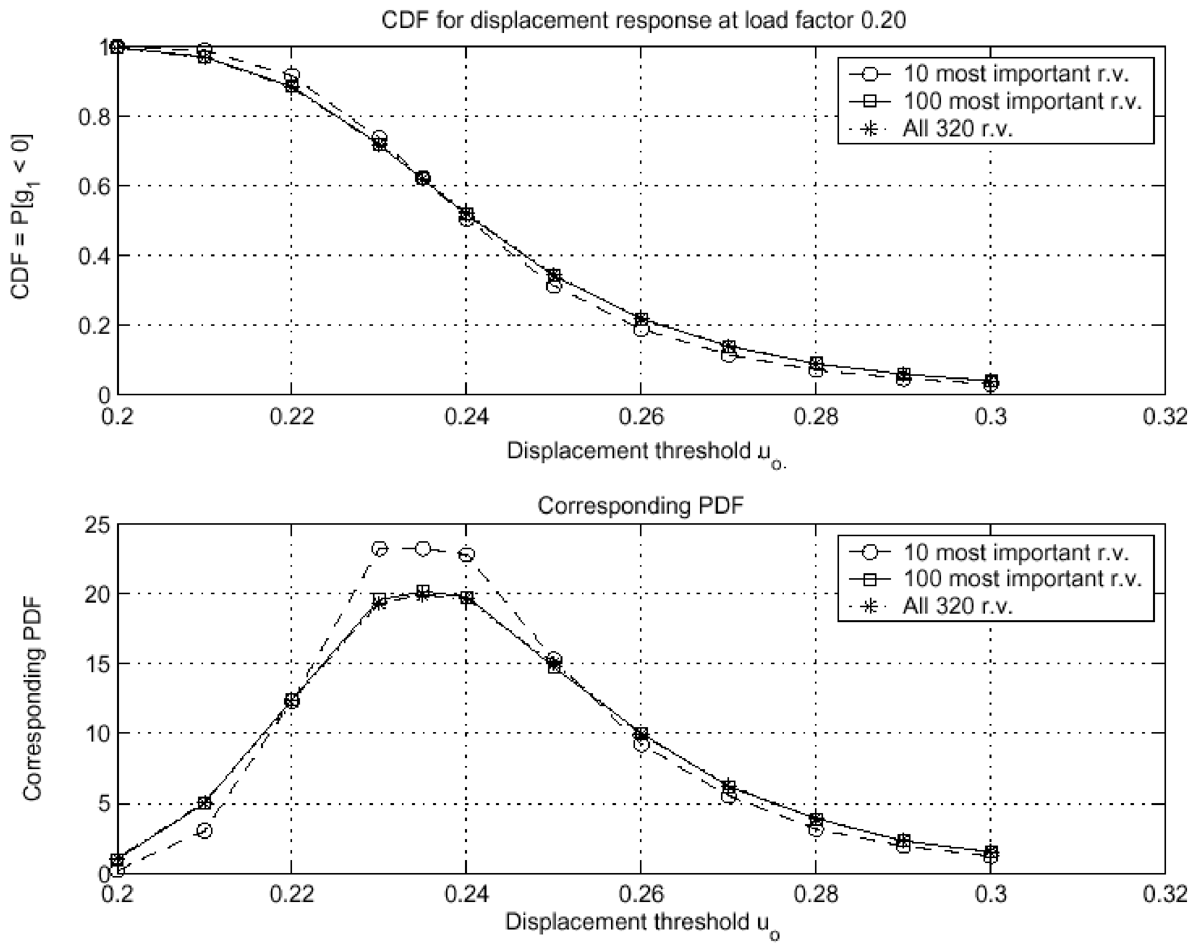


$$g_1 = u_o - u(\lambda_o)$$
$$g_2 = \lambda_o - \lambda(u_o)$$
$$g_3 = u(20\%\ tangent) - u_o$$
$$g_4 = \lambda(20\%\ tangent) - \lambda_o$$

First-Order Second-Moment Analysis

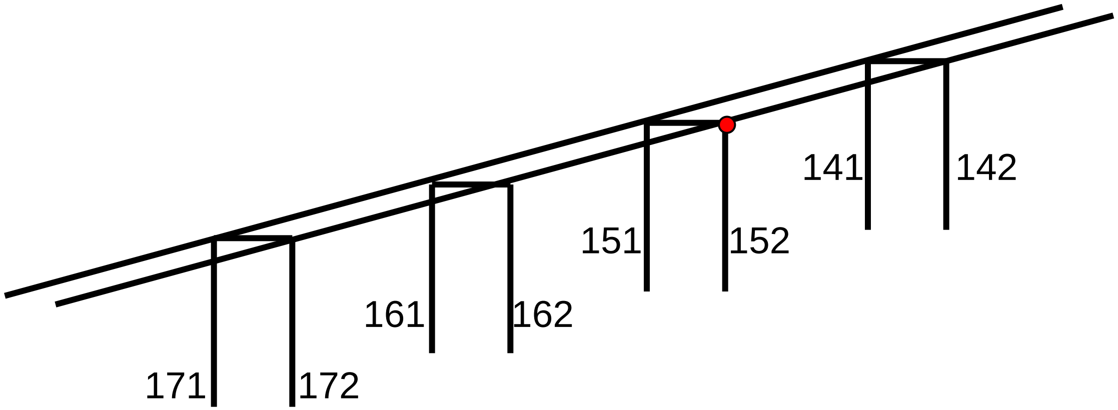


FORM Analysis, g_1, λ_o=0.20



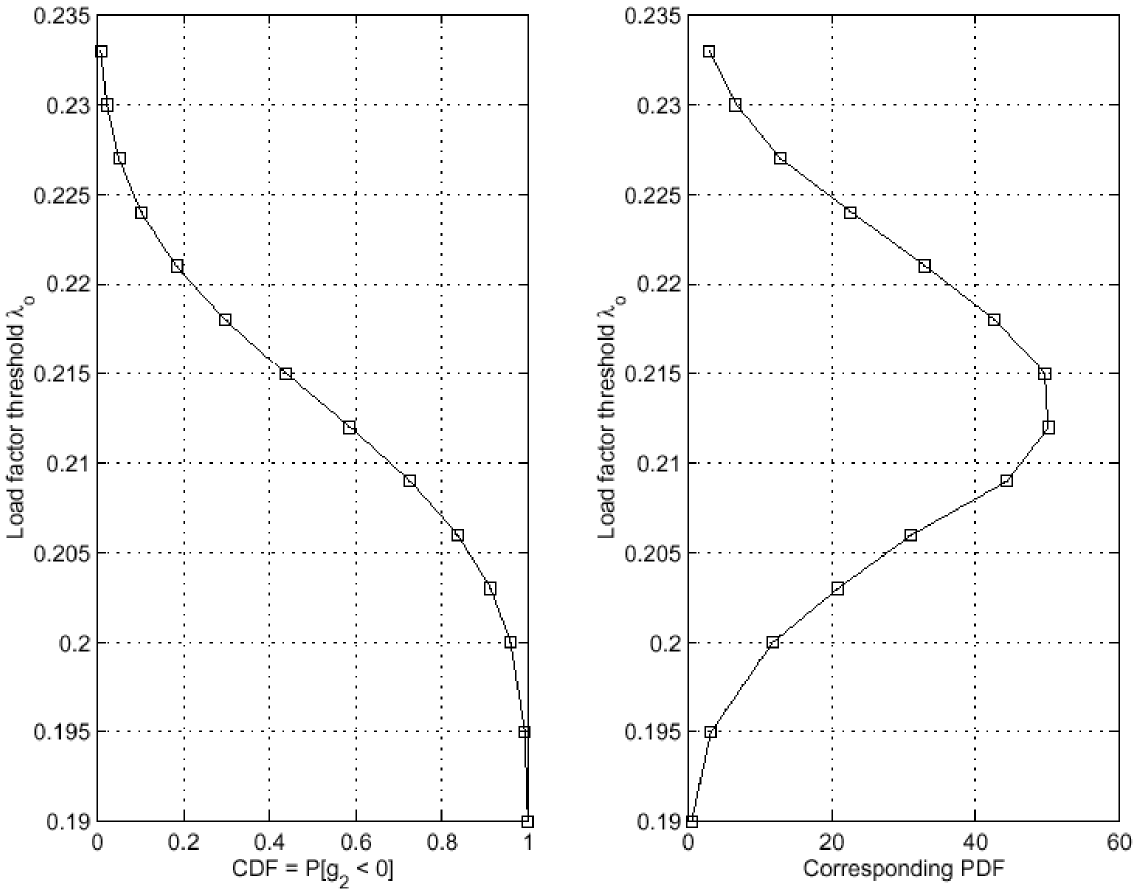
Parameter Importance

$$g_1 = 0.35 - u(\lambda{=}0.2)$$

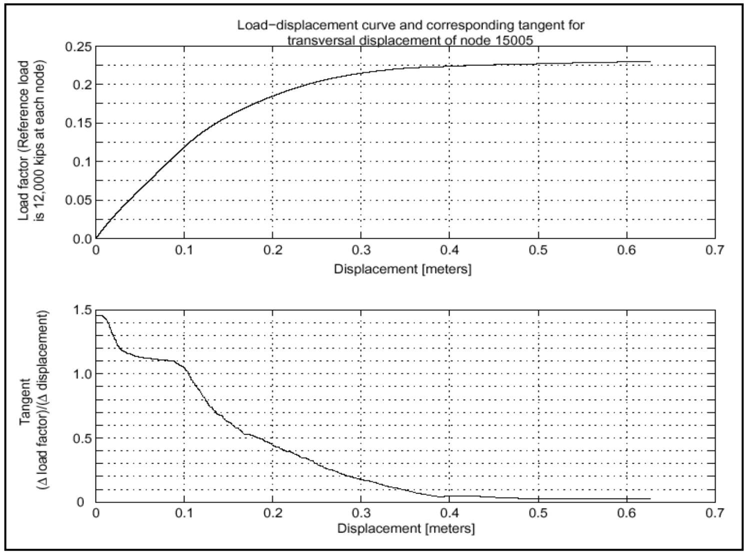


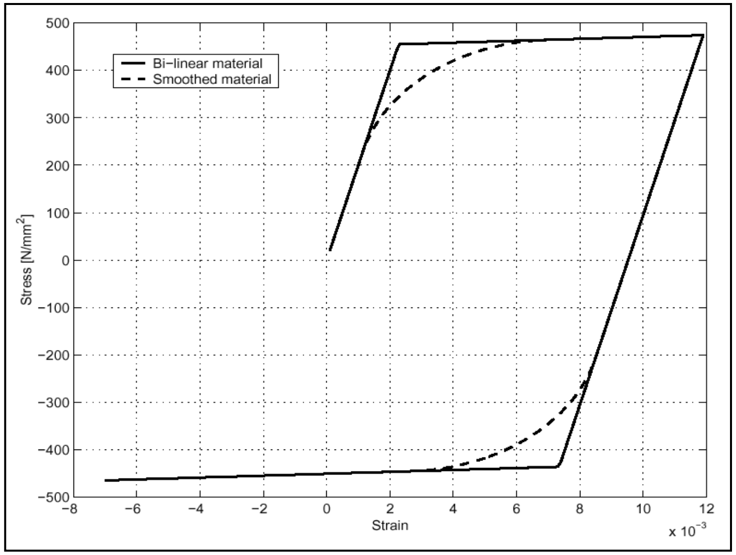
1	-0.603	Element	141	σ_y
2	-0.538	Element	142	σ_y
3	-0.280	Element	151	σ_y
4	0.240	Element	142	f'_c
5	0.232	Element	142	ε_{cu}
6	-0.188	Element	152	σ_y
7	-0.177	Element	1502	E
8	0.135	Element	142	f'_c
9	-0.122	Element	1602	E
10	-0.100	Element	161	σ_y
11	0.091	Element	141	f'_c
12	0.083	Element	152	f'_c
13	-0.073	Element	141	b
14	-0.058	Element	142	ε_c
15	-0.056	Element	162	σ_y
16	-0.048	Element	142	b
17	0.046	Element	142	ε_c
18	0.040	Element	152	ε_{cu}
19	-0.040	Element	1502	E
20	0.040	Element	152	f'_c
21	-0.032	Element	141	E
22	0.031	Element	162	f'_c
23	0.029	Element	151	f'_c
24	-0.027	Node	14002	y-crd.
25	0.026	Element	141	ε_c
26	0.026	Node	14005	y-crd.
27	-0.023	Element	1602	E
28	-0.022	Element	142	E
29	-0.022	Element	151	b
30	0.021	Element	162	ε_c

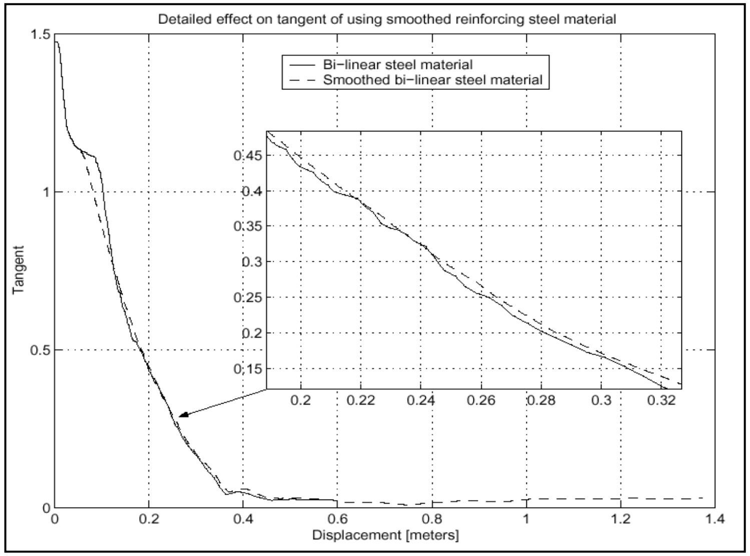
FORM Analysis, g_2, u_o=0.30

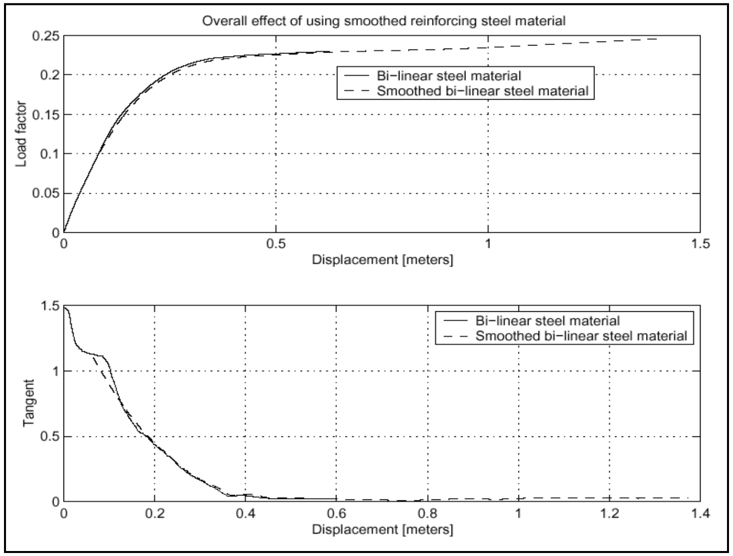


Continuity of response derivative



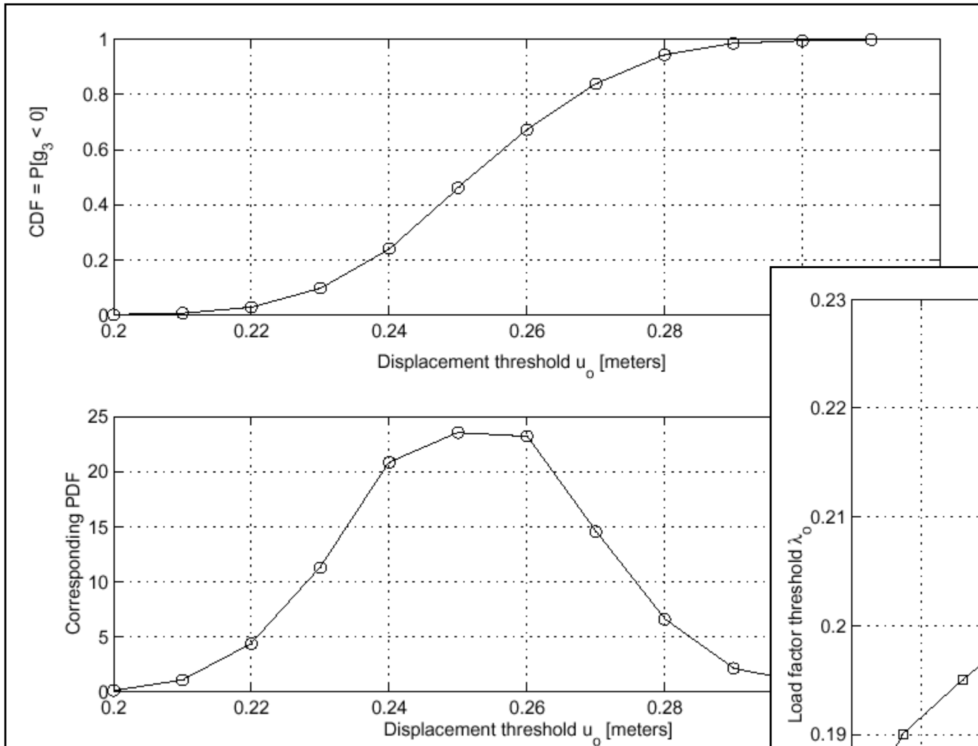


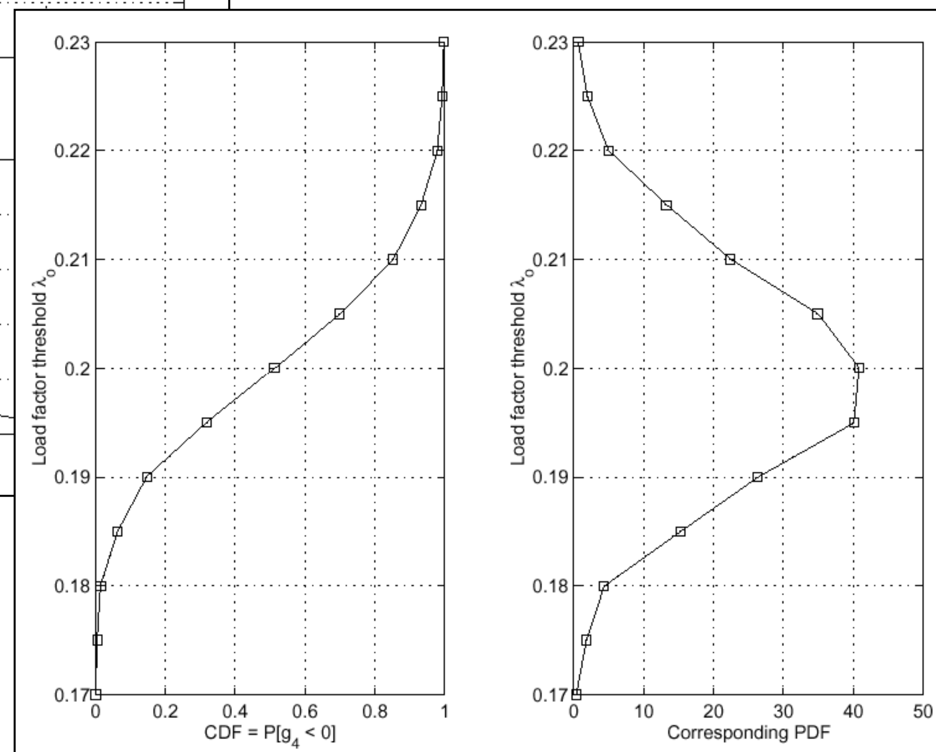




FORM Analysis, g_3 and g_4

$$g_3 = u(20\%\ tangent) - u_o$$
$$g_4 = \lambda(20\%\ tangent) - \lambda_o$$





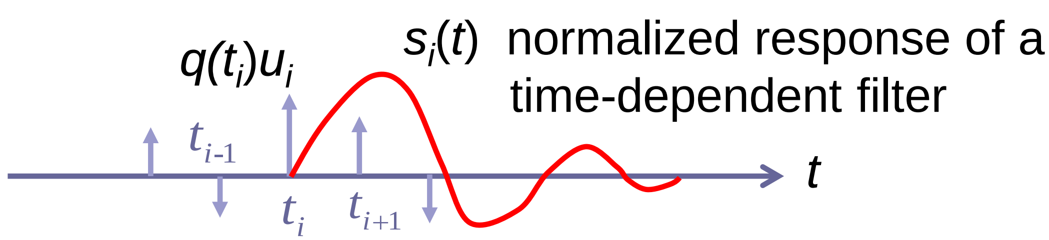
Haukaas, T., and A. Der Kiureghian (2007). Methods and object-oriented software for FE reliability and sensitivity analysis with application to a bridge structure. *Journal of Computing in Civil Engineering*, ASCE, **21**(3):151-163.

Stochastic nonlinear dynamic analysis

Representation of stochastic ground motion

$$A(t,\mathbf{u}) = q(t)\sum_{i=1}^{n} s_i(t)u_i = q(t)\,\mathbf{s}(t)\,\mathbf{u}$$

- $\mathbf{u}$: Stochasticity
- $q(t)$: Temporal nonstationarity
- $\mathbf{s}(t)$: Spectral nonstationarity



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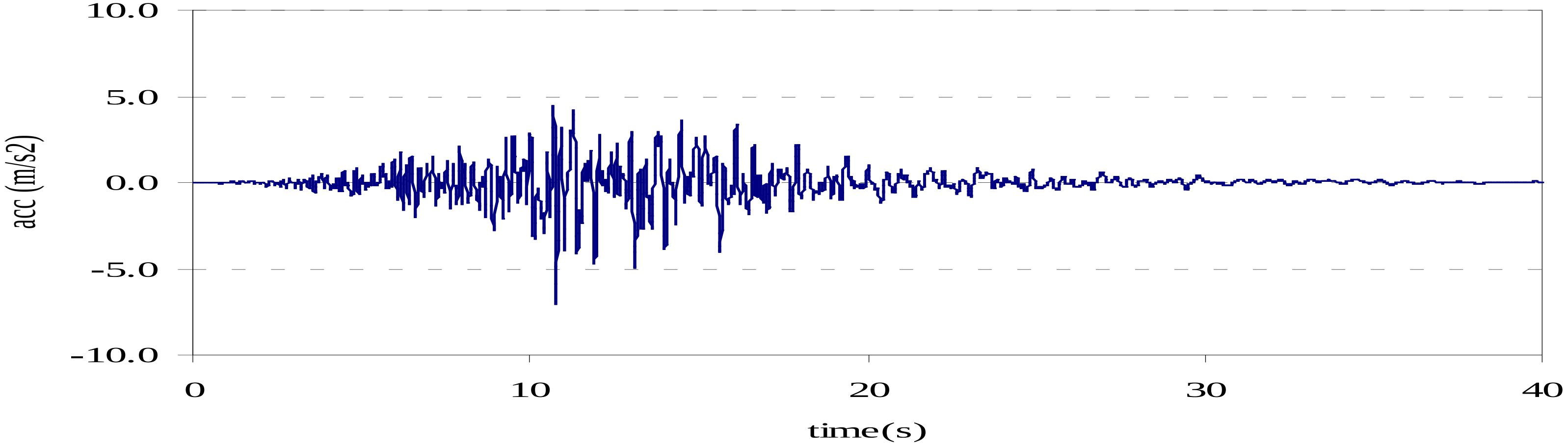


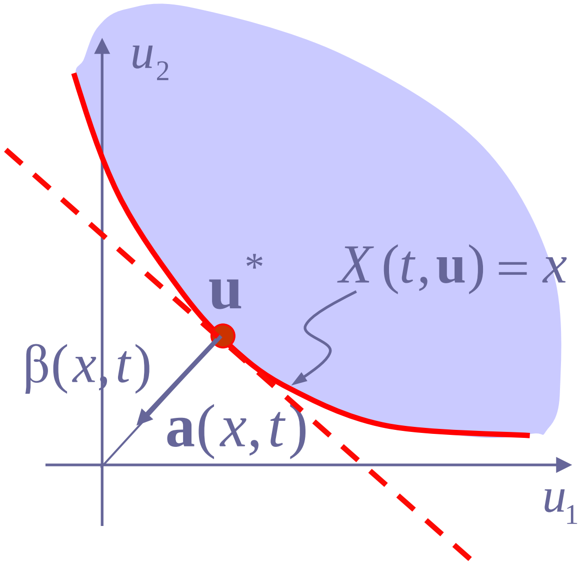
Rezaeian, S. and A. Der Kiureghian (2009). Simulation of synthetic ground motions for specified earthquake and site characteristics. *Earthquake Engineering & Structural Dynamics,* 39:1155-1180.

Stochastic nonlinear dynamic analysis

$\Pr\left[x \le X(t,\mathbf{u})\right]$ tail probability for threshold x at time t

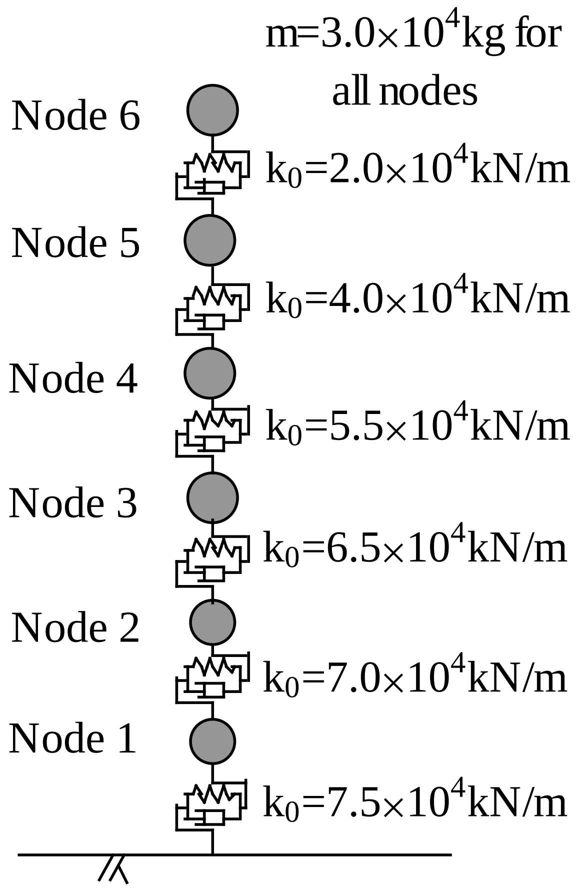
Solution of the above problem by FORM leads to identification of a Tail-Equivalent Linear System (TELS).

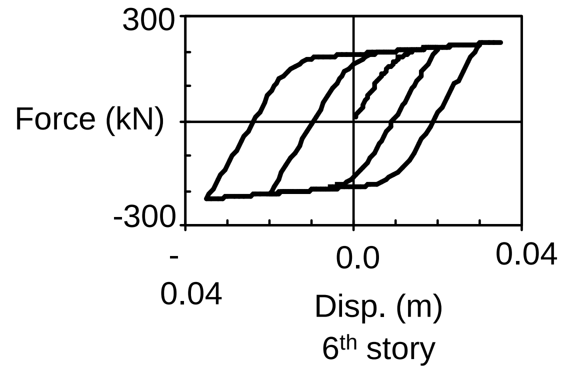
TELS is solved by linear random vibration methods to obtain response statistics of interest, e.g., distribution of extreme peak response, fragility curve.

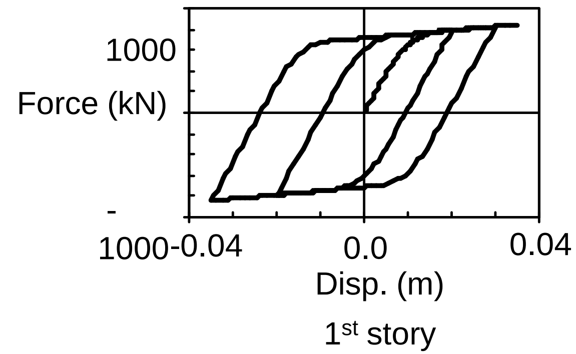


Fujimura, K., and A. Der Kiureghian (2007). Tail-equivalent linearization method for nonlinear random vibration. *Probabilistic Engineering Mechanics*, **22**:63-76.

Application to MDOF hysteretic system

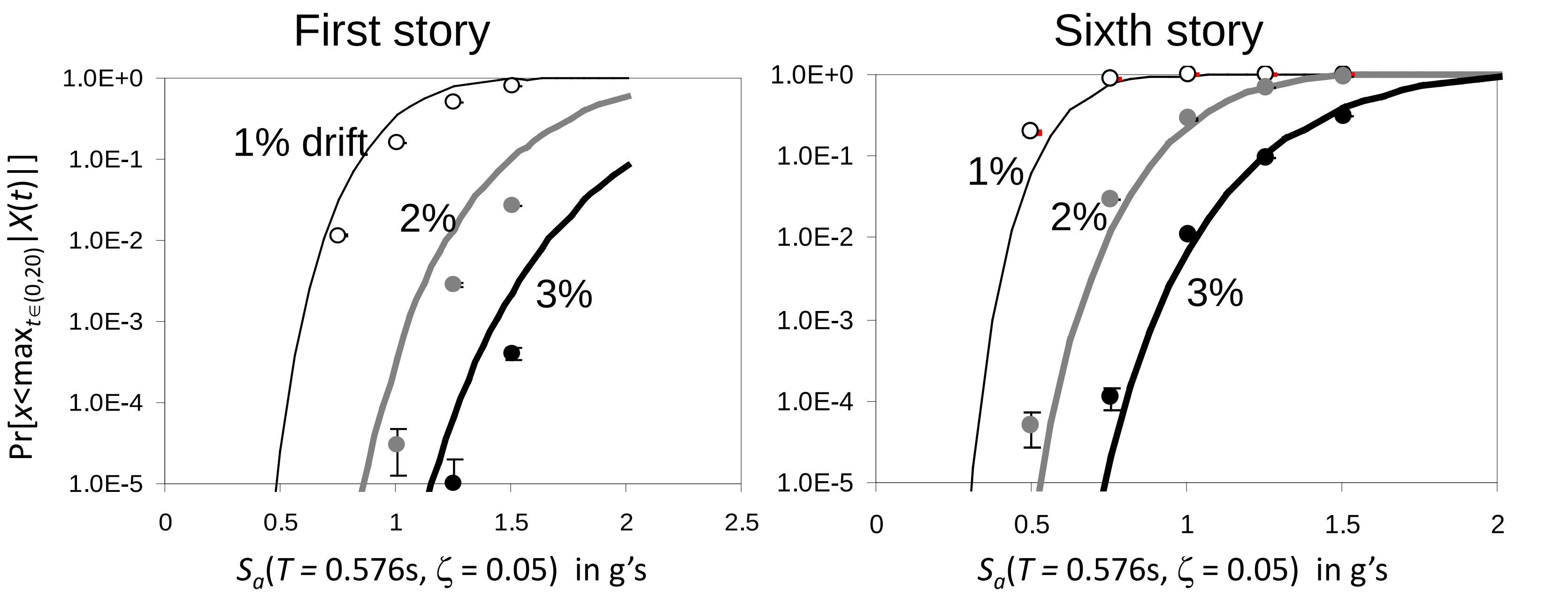






Smooth bilinear hysteresis model

Fragility curves for story drifts



Der Kiureghian, A., and K. Fujimura (2009). Nonlinear stochastic dynamic analysis for performance-based earthquake engineering*. Earthquake Engineering and Structural Dynamics*, **38**:719-738.

Summary and conclusions

- Reliability analysis requires specification of uncertain quantities, their distributions, and definition of performance via limit-state function(s). It provides probability of exceeding specified performance limit(s).
- Uncertainty propagation provides first-order approximation of mean, variance and correlations of response quantities.
- Sensitivity and importance measures provide insight into the relative importance of variables and parameters.
- Stochastic dynamic analysis is performed via tail-equivalent linearization. TELM can be used to generate fragility functions (e.g., in lieu of performing IDAs).