The OpenSees Plate Element

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1 Plate Assumptions

The first two general assumptions of plate theory are:

- The domain Ω has the following special form :

$$\Omega := \left\{ (x_1, x_2, z) \in \mathbb{R}^3 \text{ such that } z \in \left[-\frac{h}{2}, \frac{h}{2} \right] \text{ and } (x_1, x_2) \in A \subset \mathbb{R}^2 \right\}. \tag{1}$$

- The plane stress hypothesis: $\sigma_{33} = 0$.

The kinematic assumptions follow in the next section. In all future discussions, any Greek indices $\alpha, \beta, ...$ range from 1 to 2 only.

2 Kinematics

2.1 Displacement Field

The three dimensional displacement field u_i is assumed to be defined by

$$\left. \begin{array}{lcl} u_1 & := & -z\theta_1(x_1, x_2) + \bar{u}_1(x_1, x_2) \\ u_2 & := & -z\theta_2(x_1, x_2) + \bar{u}_2(x_1, x_2) \\ u_3 & := & \bar{u}_3(x_1, x_2) \end{array} \right\}, \tag{2}$$

where $\bar{u}_i$ is the translation of the plate mid-surface and θ_i are rotations of fibers initially normal to the mid-surface of the plate.

2.2 Right-Hand Rule

"Right-hand-rule" rotations $\{\hat{\theta}_1, \hat{\theta}_2\}$ are defined by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix}. \tag{3}$$

In most structural analysis codes, including OpenSees, the right-hand-rule convention is adopted. For actual numerical implementations, $\{\hat{\theta}_1, \hat{\theta}_2\}$ are used as the primary solution variables. However, it is preferable to develop the theory in terms of $\{\theta_1, \theta_2\}$, as the algebra is much simpler.

2.3 Strains

Define the curvature tensor as

$$\kappa_{\alpha\beta} := \frac{1}{2}(\theta_{\alpha,\beta} + \theta_{\beta,\alpha}), \tag{4}$$

the membrane strain tensor as

$$\bar{\epsilon}_{\alpha\beta} := \frac{1}{2}(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha}), \tag{5}$$

and the transverse shear strains as

$$\gamma_\alpha := -\theta_\alpha + \bar{u}_{3,\alpha}. \tag{6}$$

Then, the physical strains are

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) = -z\kappa_{\alpha\beta} + \bar{\epsilon}_{\alpha\beta} \tag{7}$$

and

$$\epsilon_{\alpha 3} = \epsilon_{3\alpha} = \frac{1}{2}(u_{\alpha,3} + u_{3,\alpha}) = \frac{1}{2}\gamma_\alpha. \tag{8}$$

3 Stress Resultants

3.1 Membrane

Definte the membrane stress resultant tensor as

$$p_{\alpha\beta} := \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \, \mathrm{d}z. \tag{9}$$

3.2 Bending

Define the moment tensor as

$$m_{\alpha\beta} := \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{\alpha\beta} \, \mathrm{d}z. \tag{10}$$

3.3 Shear

Define the transverse shear forces as

$$q_\alpha := \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3} \, \mathrm{d}z. \tag{11}$$

4 Constitutive Relationships

4.1 Continuum Plane Stress

Assume a linear elastic three-dimensional continuum constitutive response is given by

$$\sigma_{ij} = \mathfrak{C}_{ijkl} \cdot \epsilon_{kl}, \tag{12}$$

where $\mathfrak{C}$ is the symmetric (major and minor) rank-four elasticity tensor. Enforcement of the plane stress condition $\sigma_{33} = 0$ yields a condensed elasticity tensor $\mathbb{C}$ such that

$$\left.\begin{array}{rcl} \sigma_{ij} & = & \mathbb{C}_{ijkl} \cdot \epsilon_{kl} \\ \mathbb{C}_{ijkl} & = & \mathfrak{C}_{ijkl} - \mathfrak{C}_{ij33}(\mathfrak{C}_{3333})^{-1}\mathfrak{C}_{33kl} \end{array}\right\}. \tag{13}$$

The modified tensor $\mathbb{C}$ is now appropriate for plate analysis.

4.2 Stress Resultants

Integration through the thickness yields the stress resultant constitutive response parameters.

$$
\begin{aligned}
p_{\alpha\beta} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\ \mathrm{d}z & (14)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha\beta kl} \cdot \epsilon_{kl}\ \mathrm{d}z & (15)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbb{C}_{\alpha\beta\delta\gamma} \cdot \epsilon_{\delta\gamma} \ + \ \mathbb{C}_{\alpha\beta\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (16)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbb{C}_{\alpha\beta\delta\gamma} \cdot (-z\kappa_{\delta\gamma} + \bar{\epsilon}_{\delta\gamma}) \ + \ \mathbb{C}_{\alpha\beta\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (17)\\
&= \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} -z\mathbb{C}_{\alpha\beta\delta\gamma}\ \mathrm{d}z\right] \cdot \kappa_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha\beta\delta\gamma}\ \mathrm{d}z\right] \cdot \bar{\epsilon}_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha\beta\delta 3}\ \mathrm{d}z\right] \cdot \gamma_{\delta} & (18)
\end{aligned}
$$

$$
\begin{aligned}
m_{\alpha\beta} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{\alpha\beta}\ \mathrm{d}z & (19)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} z\mathbb{C}_{\alpha\beta kl} \cdot \epsilon_{kl}\ \mathrm{d}z & (20)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} z\left[\mathbb{C}_{\alpha\beta\delta\gamma} \cdot \epsilon_{\delta\gamma} \ + \ \mathbb{C}_{\alpha\beta\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (21)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} z\left[\mathbb{C}_{\alpha\beta\delta\gamma} \cdot (-z\kappa_{\delta\gamma} + \bar{\epsilon}_{\delta\gamma}) \ + \ \mathbb{C}_{\alpha\beta\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (22)\\
&= \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} -z^2\mathbb{C}_{\alpha\beta\delta\gamma}\ \mathrm{d}z\right] \cdot \kappa_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} z\mathbb{C}_{\alpha\beta\delta\gamma}\ \mathrm{d}z\right] \cdot \bar{\epsilon}_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} z\mathbb{C}_{\alpha\beta\delta 3}\ \mathrm{d}z\right] \cdot \gamma_{\delta} & (23)
\end{aligned}
$$

$$
\begin{aligned}
q_{\alpha} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3}\ \mathrm{d}z & (24)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha 3 kl} \cdot \epsilon_{kl}\ \mathrm{d}z & (25)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbb{C}_{\alpha 3\delta\gamma} \cdot \epsilon_{\delta\gamma} \ + \ \mathbb{C}_{\alpha 3\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (26)\\
&= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbb{C}_{\alpha 3\delta\gamma} \cdot (-z\kappa_{\delta\gamma} + \bar{\epsilon}_{\delta\gamma}) \ + \ \mathbb{C}_{\alpha 3\delta 3} \cdot 2\epsilon_{\delta 3}\right]\ \mathrm{d}z & (27)\\
&= \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} -z\mathbb{C}_{\alpha 3\delta\gamma}\ \mathrm{d}z\right] \cdot \kappa_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha 3\delta\gamma}\ \mathrm{d}z\right] \cdot \bar{\epsilon}_{\delta\gamma} \ + \ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha 3\delta 3}\ \mathrm{d}z\right] \cdot \gamma_{\delta} & (28)
\end{aligned}
$$

4.3 Isotropic Linear Elasticity

It is often convenient to use a condensed vector notation for the equations of structural mechanics. Towards that end, let

$$
\mathbf{p} := \begin{bmatrix} p_{11} \\ p_{22} \\ p_{12} \end{bmatrix}, \ \mathbf{m} := \begin{bmatrix} m_{11} \\ m_{22} \\ m_{12} \end{bmatrix}, \ \mathbf{q} := \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad (29)
$$

and

$$\bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ 2\bar{\epsilon}_{12} \end{bmatrix}, \; \boldsymbol{\kappa} = \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{bmatrix}, \; \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}. \tag{30}$$

Let $M := \frac{Eh}{1-\nu^2}$ be the membrane modulus, $G := \frac{Eh}{2(1+\nu)}$ be the shear modulus, $D := \frac{Eh^3}{12(1-\nu^2)}$ be the bending modulus and $k := \frac{5}{6}$ be the shear correction factor. The constitutive response of a linear elastic isotropic plate is

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{m} \\ \mathbf{q} \end{bmatrix} = \underbrace{\begin{bmatrix} M & \nu M & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu M & M & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & -\nu D & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu D & -D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(1-\nu)D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & kG & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & kG \end{bmatrix}}_{\mathbb{D}} \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{bmatrix}. \tag{31}$$

General elastic constitutive models have the form

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{m} \\ \mathbf{q} \end{bmatrix} = \mathbb{D} \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{bmatrix}, \tag{32}$$

where $\mathbb{D}$ is a symmetric positive-definite 8×8 matrix.

5 Principle of Virtual Work

Let $\{\delta\bar{u}_i, \delta\theta_i\}$ be kinematically admissable variations of mid-surface displacment and rotation, repectively. Corresponding to these are the variations $\{\delta\bar{\boldsymbol{\epsilon}}, \delta\boldsymbol{\kappa}, \delta\boldsymbol{\gamma}\}$:

$$\delta\bar{\epsilon}_{\alpha\beta} := \frac{1}{2}(\delta\bar{u}_{\alpha,\beta} + \delta\bar{u}_{\beta,\alpha}), \tag{33}$$

$$\delta\kappa_{\alpha\beta} := \frac{1}{2}(\delta\theta_{\alpha,\beta} + \delta\theta_{\beta,\alpha}), \tag{34}$$

$$\delta\gamma_\alpha := -\delta\theta_\alpha + \delta\bar{u}_{3,\alpha}. \tag{35}$$

Assume the plate is loaded by external forces F_i acting per unit area. For simplicity, boundary loading terms and distributed external moments are omitted. Starting from the basic equation

$$\delta\Pi := \int_\Omega \delta\epsilon_{ij} \cdot \sigma_{ij} \, \mathrm{d}\Omega \; - \; \int_A \delta\bar{u}_i \cdot F_i \, \mathrm{d}A \; = \; 0 \qquad \forall\{\delta\bar{u}_i, \delta\theta_i\}, \tag{36}$$

the weak form of equilibrium for a plate can be derived. Substituting the plate kinematics and stress resultant definitions produces

$$\int_\Omega \delta\epsilon_{ij}\cdot\sigma_{ij}\ \mathrm{d}\Omega \;=\; \int_A \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \delta\epsilon_{ij}\cdot\sigma_{ij}\ \mathrm{d}z\right]\mathrm{d}\Omega \tag{37}$$

$$= \int_A \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (-z\delta\kappa_{\alpha\beta}+\delta\bar{\epsilon}_{\alpha\beta})\cdot\sigma_{\alpha\beta} \;+\; 2\delta\epsilon_{\alpha 3}\cdot\sigma_{\alpha 3}\ \mathrm{d}z\right]\mathrm{d}A \tag{38}$$

$$= \int_A -\delta\kappa_{\alpha\beta}\cdot\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{\alpha\beta}\ \mathrm{d}z\right]\mathrm{d}A \;+\; \int_A \delta\bar{\epsilon}_{\alpha\beta}\cdot\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}\ \mathrm{d}z\right]\mathrm{d}A$$

$$+\int_A \delta\gamma_\alpha\cdot\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3}\ \mathrm{d}z\right]\mathrm{d}A \tag{39}$$

$$= \int_A (-\delta\kappa_{\alpha\beta}\cdot m_{\alpha\beta} \;+\; \delta\gamma_\alpha\cdot q_\alpha \;+\; \delta\bar{\epsilon}_{\alpha\beta}\cdot p_{\alpha\beta})\ \mathrm{d}A. \tag{40}$$

Thus

$$\delta\Pi = \int_A (-\delta\kappa_{\alpha\beta}\cdot m_{\alpha\beta} \;+\; \delta\gamma_\alpha\cdot q_\alpha \;+\; \delta\bar{\epsilon}_{\alpha\beta}\cdot p_{\alpha\beta})\ \mathrm{d}A \;-\; \int_A \delta\bar{u}_i\cdot F_i\ \mathrm{d}A \;=\; 0 \qquad \forall\{\delta\bar{u}_i,\delta\theta_i\}. \tag{41}$$

This may also be written in vector notation as

$$\delta\Pi = \int_A \left(-\delta\boldsymbol{\kappa}^T\mathbf{m} \;+\; \delta\boldsymbol{\gamma}^T\mathbf{q} \;+\; \delta\bar{\boldsymbol{\epsilon}}^T\mathbf{p}\right)\ \mathrm{d}A \;-\; \int_A \delta\bar{\mathbf{u}}^T\mathbf{F}\ \mathrm{d}A \;=\; 0 \qquad \forall\{\delta\bar{\mathbf{u}},\delta\boldsymbol{\theta}\}. \tag{42}$$

6 "Drilling" Degree-of-Freedom

Let $\hat{\theta}_3$ be the "drilling" rotation, a degree-of-freedom representing rotations in the mid-surface plane of the plate. Let $k_{\theta\theta} > 0$ be the "drilling stiffness" (penalty parameter). Define

$$\Omega(\bar{\mathbf{u}}) := \frac{1}{2}(-\bar{u}_{1,2}+\bar{u}_{2,1}) \tag{43}$$

as the in-plane rotation of the plate. The goal here is to weakly enforce the condition

$$\hat{\theta}_3 - \Omega(\bar{\mathbf{u}}) = 0. \tag{44}$$

Define the "drilling energy" as

$$\Pi_{drill} := \frac{1}{2}\int_A k_{\theta\theta}\left[\hat{\theta}_3 - \Omega(\bar{\mathbf{u}})\right]^2\ \mathrm{d}A \tag{45}$$

6.1 Weak Form

Let $\delta\hat{\theta}_3$ be a kinematically adimissable variation of $\hat{\theta}_3$. To $\delta\Pi$ is added

$$\delta\Pi_{drill} = \int_A \left[\delta\hat{\theta}_3 - \Omega(\delta\bar{\mathbf{u}})\right]\cdot k_{\theta\theta}\left[\hat{\theta}_3 - \Omega(\bar{\mathbf{u}})\right]\ \mathrm{d}A = 0 \qquad \forall\{\delta\bar{\mathbf{u}},\delta\hat{\theta}_3\}. \tag{46}$$

An appropriate choice for $k_{\theta\theta}$ is

$$k_{\theta\theta} := \min\ \text{eigenvalue}\ \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbb{C}_{\alpha\beta\delta\gamma}\ \mathrm{d}z\right]. \tag{47}$$

For isotropic linear elasticity typically $k_{\theta\theta} = G$ as previously defined. It is *not* necessary for $k_{\theta\theta}$ to be taken to infinity. Notice that one of the Euler equations of $\delta\Pi_{drill}$ above is

$$\int_A \delta\hat{\theta}_3\cdot k_{\theta\theta}\left[\hat{\theta}_3 - \Omega(\bar{\mathbf{u}})\right]\ \mathrm{d}A = 0\ \ \forall\hat{\theta}_3 \iff k_{\theta\theta}\left[\hat{\theta}_3 - \Omega(\bar{\mathbf{u}})\right] = 0 \iff \hat{\theta}_3 - \Omega(\bar{\mathbf{u}}) = 0. \tag{48}$$

The weak form is variationally consistent independent of the choice of $k_{\theta\theta} > 0$.

7 Standard Plate Finite Element

Consistent with vector notation, define

$$\bar{\mathbf{u}} := \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} \tag{49}$$

and

$$\hat{\boldsymbol{\theta}} := \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{bmatrix}. \tag{50}$$

Now, consider a standard four-node plate finite element. The domain of this element shall be denoted by $A_e \subset A$. The displacement and rotation interpolations are given by

$$\bar{\mathbf{u}}(\xi, \eta) = \sum_{I=1}^{4} N_I(\xi, \eta)\bar{\mathbf{u}}^I \tag{51}$$

and

$$\hat{\boldsymbol{\theta}}(\xi, \eta) = \sum_{I=1}^{4} N_I(\xi, \eta)\hat{\boldsymbol{\theta}}^I. \tag{52}$$

In this setting $\bar{\mathbf{u}}^I$ and $\hat{\boldsymbol{\theta}}^I$, $I = \{1, 2, 3, 4\}$, are the nodal values of the mid-surface displacement and the rotation, respectively. The functions $N_I(\xi, \eta)$ are the standard bilinear isoparametric shape functions, defined in terms of the natural coordinates $(\xi, \eta) \in [-1, 1] \times [-1, 1]$.

These equations produce, in standard fashion, "strain-displacement" matrices. These are constructed such that

$$\bar{\boldsymbol{\epsilon}} = \sum_{I=1}^{4} \mathbf{B}_I^M(\xi, \eta)\bar{\mathbf{u}}^I, \tag{53}$$

$$\boldsymbol{\kappa} = \sum_{I=1}^{4} \mathbf{B}_I^B(\xi, \eta)\hat{\boldsymbol{\theta}}^I, \tag{54}$$

and

$$\boldsymbol{\gamma} = \sum_{I=1}^{4} \mathbf{B}_I^S(\xi, \eta)\begin{bmatrix} \bar{\mathbf{u}}^I & \hat{\boldsymbol{\theta}}^I \end{bmatrix}^T. \tag{55}$$

The admissible variations $\{\delta\bar{\boldsymbol{\epsilon}}, \delta\boldsymbol{\kappa}, \delta\boldsymbol{\gamma}\}$ are interpolated in the same manner. The details of constructing these matrices are well known and are omitted here.

A combined "strain-displacement" matrix may be defined such that

$$\begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{bmatrix} = \sum_{I=1}^{4} \underbrace{\begin{bmatrix} \mathbf{B}_I^M & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_I^B \\ \multicolumn{2}{c}{\mathbf{B}_I^S} \end{bmatrix}}_{\mathbf{B}_I} \begin{bmatrix} \bar{\mathbf{u}}^I \\ \hat{\boldsymbol{\theta}}^I \end{bmatrix} \tag{56}$$

At the element level, the principle of virtual work (weak form) can be written as

$$\delta\Pi_e = \sum_{I=1}^{4} \begin{bmatrix} \delta\bar{\mathbf{u}}^I \\ \delta\hat{\boldsymbol{\theta}}^I \end{bmatrix}^T \left[\int_{A_e} \mathbf{B}_I^T \begin{bmatrix} \mathbf{p} \\ \mathbf{m} \\ \mathbf{q} \end{bmatrix} \mathrm{d}A_e \right]. \tag{57}$$

The element "stiffness" matrix linking nodes I and J is

$$\mathbf{K}_{IJ} = \int_{A_e} \mathbf{B}_I^T \mathbb{D} \mathbf{B}_J \; \mathrm{d}A_e. \tag{58}$$

Note also that a "drilling strain-displacement" matrix is defined such that

$$\hat{\theta}_3 - \Omega(\bar{\mathbf{u}}) = \sum_{I=1}^{4} \mathbf{B}_I^{dr} \left[\begin{array}{cc} \bar{\mathbf{u}}^I & \hat{\boldsymbol{\theta}}^I \end{array} \right]^T. \tag{59}$$

This produces an additional "drilling stiffness" matrix

$$\mathbf{K}_{IJ}^{dr} = \int_{A_e} \mathbf{B}_I^{dr\,T} k_{\theta\theta} \mathbf{B}_J^{dr} \, \mathrm{d}A_e. \tag{60}$$

8 "B-Bar" Plate Element Formulation

It is well known that the standard plate element formulation performs poorly within the context of thin plates. The element exhibits an overly "stiff" behavior commonly referred to as "shear locking."

As an example, consider a simply supported plate where $A = [0,1] \times [0,1]$. Assume this domain is discretized by standard quadrilateral plate finite elements. The plate is centrally loaded by a point load F at the center point $(x_1, x_2) = (0.5, 0.5)$ of the plate. Let Δ be the displacement of the node under the point load at the point $(0.5, 0.5)$. Then, as the thickness h goes to zero,

$$\lim_{h \to 0} \Delta(h) = 0. \tag{61}$$

This is of course a completely nonphysical result. As the plate becomes thin, the shear strains vanish but the plate should still respond in bending when loaded. From a finite element perspective, one possible solution to this problem is a *re-definition* of the shear "strain-displacement" matrices $\mathbf{B}_I^S$.

Define the Jacobian matrix of the isoparametric map as

$$\mathbf{J}(\xi, \eta) := \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi} & \dfrac{\partial x_1}{\partial \eta} \\ \\ \dfrac{\partial x_2}{\partial \xi} & \dfrac{\partial x_2}{\partial \eta} \end{bmatrix}, \tag{62}$$

and let $\mathbf{J}_0 := \mathbf{J}(0,0)$. Additionally, define

$$\mathbf{e}_\xi := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_\eta := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{63}$$

The modified "B-bar" shear strain $\bar{\boldsymbol{\gamma}}$ is computed by the following steps:

1. Compute the four parent element natural coordinate shear strain parameters:

$$\left. \begin{array}{rcl} \gamma_1^B & := & \mathbf{e}_\xi^T \mathbf{J}_0^T \, \boldsymbol{\gamma}(0,-1) \\ \gamma_1^D & := & \mathbf{e}_\xi^T \mathbf{J}_0^T \, \boldsymbol{\gamma}(0,1) \\ \gamma_2^A & := & \mathbf{e}_\eta^T \mathbf{J}_0^T \, \boldsymbol{\gamma}(-1,0) \\ \gamma_2^C & := & \mathbf{e}_\eta^T \mathbf{J}_0^T \, \boldsymbol{\gamma}(1,0) \end{array} \right\}. \tag{64}$$

   In the above, $\boldsymbol{\gamma}(\xi, \eta)$ is computed using the standard $\mathbf{B}_I^S$ matrices. For example,

$$\boldsymbol{\gamma}(0,-1) = \sum_{I=1}^{4} \mathbf{B}_I^S(0,-1) \left[\begin{array}{cc} \bar{\mathbf{u}}^I & \hat{\boldsymbol{\theta}}^I \end{array} \right]^T. \tag{65}$$

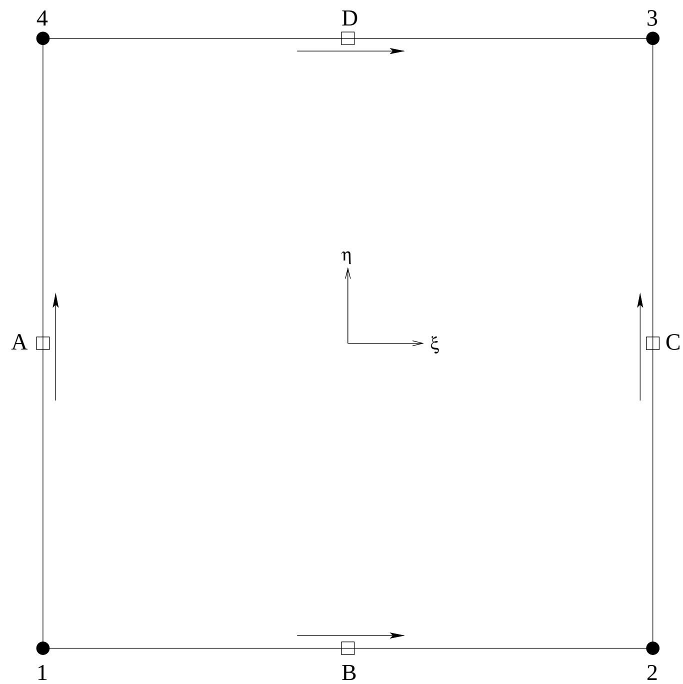

Figure 1: Transverse Shear Strain Collocation Points

2. Interpolate to define the parent element shear strain distribution:

$$\boldsymbol{\gamma}_{\square} := \begin{bmatrix} \gamma_\xi \\ \gamma_\eta \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (1-\eta)\gamma_1^B + (1+\eta)\gamma_1^D \\ (1-\xi)\gamma_2^A + (1+\xi)\gamma_2^C \end{bmatrix}. \tag{66}$$

3. Transform the shear strain back to the physical coordinates to define a shear strain $\boldsymbol{\gamma}_*$:

$$\boldsymbol{\gamma}_* := \mathbf{J}_0^{-T} \boldsymbol{\gamma}_{\square}. \tag{67}$$

4. Apply an integral average to define $\bar{\boldsymbol{\gamma}}$:

$$\bar{\boldsymbol{\gamma}} := \boldsymbol{\gamma}_* + \frac{1}{\text{meas}(A_e)} \int_{A_e} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_*) \, \mathrm{d}A_e. \tag{68}$$

   If the Jacobian is constant ($\mathbf{J}(\xi,\eta) = \mathbf{J}_0$) then $\bar{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_*$. For non-constant Jacobian element configurations, in general $\bar{\boldsymbol{\gamma}} \neq \boldsymbol{\gamma}_*$.

Modified shear "strain-displacement" matrices $\bar{\mathbf{B}}_I^S$ are computed such that

$$\bar{\boldsymbol{\gamma}} = \sum_{I=1}^{4} \bar{\mathbf{B}}_I^S(\xi,\eta) \begin{bmatrix} \mathbf{u}^I & \hat{\boldsymbol{\theta}}^I \end{bmatrix}^T. \tag{69}$$

These modified "B-bar" matrices are used in all element computations, including residual and tangent calculations. For example, a combined "B-bar strain-displacement" matrix $\bar{\mathbf{B}}_I$ may be constructed such that

$$\begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \boldsymbol{\kappa} \\ \bar{\boldsymbol{\gamma}} \end{bmatrix} = \sum_{I=1}^{4} \underbrace{\begin{bmatrix} \mathbf{B}_I^M & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_I^B \\ \multicolumn{2}{c}{\bar{\mathbf{B}}_I^S} \end{bmatrix}}_{\bar{\mathbf{B}}_I} \begin{bmatrix} \bar{\mathbf{u}}^I \\ \hat{\boldsymbol{\theta}}^I \end{bmatrix}, \tag{70}$$

producing the element "stiffness" matrix

$$\mathbf{K}_{IJ} = \int_{A_e} \bar{\mathbf{B}}_I^T \mathbb{D} \bar{\mathbf{B}}_J \, \mathrm{d}A_e. \tag{71}$$

Some important observations must be made about this formulation:

- In almost all cases this element is free of "shear locking" and performs well in both thick and thin plate analysis.
- Only the shear interpolation is modified. The membrane, bending and drilling interpolations remain unchanged.
- The presence of the Jacobian matrix $\mathbf{J}_0$ in the shear strain transformations ensures invariance of the formulation with respect to stretch and rotation of the element relative to the parent domain $[-1,1] \times [-1,1]$.
- If the shear strain $\boldsymbol{\gamma}$ computed from nodal parameters is spatially constant, then $\bar{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}_*$ are also constant and $\bar{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_* = \boldsymbol{\gamma}$. Thus constant shear deformation modes are exactly representable in this formulation. This is a consistency requirement and is necessary for the element to pass the patch test.
- The integral average of step four(4) ensures that

$$\int_{A_e} \left(\mathbf{B}_I^S - \bar{\mathbf{B}}_I^S \right) \, \mathrm{d}A_e = \mathbf{0} \tag{72}$$

  for both constant Jacobian ($\mathbf{J}(\xi, \eta) = \mathbf{J}_0$) and non-constant Jacobian element configurations. This condition is sufficient to guarantee both variational consistency of the formulation and satisfaction of additional patch test requirements.
- The element is applicable, without modifications, to non-linear material response. In addition, this modified shear strain interpolation can be extended to properly invariant geometrically exact shell theory. However, such a theory is beyond the scope of this report.

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A Shell Modifications

Consider a single flat shell finite element. Let $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ be basis vectors for the element such that $\|\mathbf{g}_i\| = 1$, $\mathbf{g}_i \cdot \mathbf{g}_j = \delta_{ij}$ and $\mathbf{g}_1 \times \mathbf{g}_2 = \mathbf{g}_3$. Thus these three vectors form a right-handed, orthonormal basis for $\mathbb{R}^3$. Assume that $\mathbf{g}_3$ is normal to the shell mid-surface so that $\mathbf{g}_1$ and $\mathbf{g}_2$ lie in the tangent plane of the shell mid-surface.

In this setting any vector $\mathbf{v} \in \mathbb{R}^3$ (position, displacement, ...) can be written as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{g}_i)\, \mathbf{g}_i. \tag{73}$$

The scalars $(\mathbf{v} \cdot \mathbf{g}_i)$ are the components of $\mathbf{v}$ relative to the$\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ coordinate system. For example, when considering the plate kinematics, the components of the mid-surface displacement vector $\bar{\mathbf{u}}$ are

$$\left.\begin{array}{rcl} \bar{u}_1 & = & \bar{\mathbf{u}} \cdot \mathbf{g}_1 \\ \bar{u}_2 & = & \bar{\mathbf{u}} \cdot \mathbf{g}_2 \\ \bar{u}_3 & = & \bar{\mathbf{u}} \cdot \mathbf{g}_3 \end{array}\right\}, \tag{74}$$

and the components of the rotation vector $\hat{\boldsymbol{\theta}}$ are

$$\left.\begin{array}{rcl} \hat{\theta}_1 & = & \hat{\boldsymbol{\theta}} \cdot \mathbf{g}_1 \\ \hat{\theta}_2 & = & \hat{\boldsymbol{\theta}} \cdot \mathbf{g}_2 \\ \hat{\theta}_3 & = & \hat{\boldsymbol{\theta}} \cdot \mathbf{g}_3 \end{array}\right\}. \tag{75}$$

Additionally, any symmetric rank-two tensor $\mathbf{T}$ (strain, curvature, ...) can be written as

$$\mathbf{T} = (\mathbf{g}_i \cdot \mathbf{T}\mathbf{g}_j)\, \mathbf{g}_i \otimes \mathbf{g}_j, \tag{76}$$

where $(\mathbf{g}_i \cdot \mathbf{T}\mathbf{g}_j)$ are the components of $\mathbf{T}$. Expressions for higher rank tensors can also be defined. Spatial partial derivatives may be calculated relative to this coordinate system.

Thus all the equations for the plate element hold in the $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ coordinate system. By performing all calculations relative to this system, and without any additional modifications to the theory, the plate element becomes a flat shell element.

B Kinetic Energy

Let $\rho > 0$ be the mass density per unit volume of the plate material. Ignoring rotational inertia terms, the kinetic energy $\mathbb{T}$ of the plate is

$$\begin{aligned} 2\mathbb{T} &:= \int_\Omega \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, \mathrm{d}\Omega & (77) \\ &= \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, \mathrm{d}z \, \mathrm{d}A & (78) \\ &= \int_A \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, \mathrm{d}z \right] \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, \mathrm{d}A & (79) \\ &=: \int_A \bar{\rho} \, \dot{\mathbf{u}}^T \dot{\mathbf{u}} \, \mathrm{d}A, & (80) \end{aligned}$$

where

$$\bar{\rho} := \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, \mathrm{d}z \tag{81}$$

is the mass per unit area of the plate. If ρ is constant through the cross-section z of the plate then $\bar{\rho} = \rho h$.

To $\delta\Pi$ is added

$$\delta\Pi_{inertia} = \int_A \delta\bar{\mathbf{u}}^T \bar{\rho} \ddot{\mathbf{u}} \, \mathrm{d}A. \tag{82}$$